A PAGE FROM NEWTON'S NOTEBOOKS

[Background. Isaac Newton's notebooks have been digitized, and are available through Cambridge University. Among them is one labeled Add. 3960. As the notes on the site

`https://cudl.lib.cam.ac.uk/view/MS-ADD-03960/1`

mention, "Many items in Add. 3960 relate to Newton's planning, drafts, publication, and revision of *Tractatus de quadratura curvarum,* a treatise devoted to what we would now call integration, which Newton composed in the early 1690s, reworked in the early 1700s, and eventually published as an appendix to the *Opticks* (1704)." This example is part of Problem 9 (p. 405): "To determine the area of various given curves." He uses the notation from Problem 8; there he has sketched some graphs $r = f(x)$ over an interval AB, where the graph leads from (A, F) to (B, D), so the corresponding area $z = \int_0^x f(t)\, dt$ is that of the curvilinear polyhedron $AFDB$; the letter H refers to a value larger than B.

In this example he is calculating areas by anti-differentiation, using his Binomial Theorem to reduce integrals involving radicals to the term-by-term integration of power series.]

Example: 4. Where a previous reduction by extraction of roots is required.

Given $r = \sqrt{aa + xx}$ (the equation of a hyperbola), extracting the root by an infinite sequence yields

$$r = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{112a^7} \quad \&\text{c.}$$

[This curve is the graph of $r = \sqrt{a^2 + x^2}$. Newton writes the root as $a\sqrt{1 + (\frac{x}{a})^2}$, and uses his binomial theorem

$$(1 + t)^{\frac{1}{2}} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \frac{1}{16}t^3 - \frac{5}{128}t^4 + \dots. \qquad (*)$$

We substitute $t = (\frac{x}{a})^2$ and multiply by a:

$$r = a + \frac{1}{2}(\frac{x^2}{a}) - \frac{1}{8}(\frac{x^4}{a^3}) + \frac{1}{16}(\frac{x^6}{a^5}) - \frac{5}{128}(\frac{x^8}{a^7}) + \dots.$$

Newton has a numerical error in the last denominator.]

Proceeding as before,

$$z = ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7} \quad \&\text{c.}$$

[Newton integrates the series term by term. The last term listed should have 1152 instead of 1008.]

In the same way if given $r = \sqrt{aa - xx}$ (the equation of a circle), we obtain

$$z = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7} \quad \&c.$$

[Again, the last term listed should have 1152 instead of 1008.]

Again if given $r = \sqrt{x - xx}$ (the equation of a circle) we obtain from extraction of the root

$$r = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} \quad \&c$$

and thus

$$z = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}} \quad \&c.$$

[Here Newton writes $r = \sqrt{x - x^2}$ as $r = x^{\frac{1}{2}}\sqrt{1-x}$ and applies the binomial theorem $(*)$ to yield a series which he integrates term by term.]

Thus $r = \sqrt{aa + bx - xx}$ (the equation of a circle) gives by extraction of roots

$$r = a + \frac{bx}{2a} - \frac{xx}{2a} - \frac{bbxx}{8a^3} \quad \&c$$

whence by Prop 2 we derive

$$z = ax + \frac{bxx}{4a} - \frac{x^3}{6a} - \frac{bbx^3}{24a^3} \quad \&c.$$

[Here Newton expands $r = \sqrt{aa + bx - xx}$ as $r = a\sqrt{1 + \frac{bx - xx}{aa}}$ using $(*)$.]

And likewise $\sqrt{\frac{1+axx}{1-bxx}} = r$, by the required reduction gives

$$r = 1 + \begin{matrix} +\frac{1}{2}b \\ +\frac{1}{2}a \end{matrix} \; x^2 \quad \begin{matrix} +\frac{3}{8}bb \\ +\frac{1}{4}ab \\ -\frac{1}{8}aa \end{matrix} \; x^4 \quad \&c.$$

[Here Newton writes $\sqrt{\frac{1+axx}{1-bxx}}$ as $(1+axx)^{1/2}(1-bxx)^{-1/2}$ and multiplies the two binomial expansions.]

Whence by Prop 2 we obtain

$$z = x + \begin{array}{l} +\frac{1}{6}b \\ +\frac{1}{6}a \end{array} x^3 \quad \begin{array}{l} +\frac{3}{40}bb \\ +\frac{1}{20}ab \\ -\frac{1}{40}aa \end{array} x^5 \quad \&\text{c.}$$

Likewise finally $r = \sqrt[3]{a^3 + x^3}$ by extraction of the cube root gives

$$r = a + \frac{x^3}{3aa} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} \quad \&\text{c}$$

and thus

$$z = ax + \frac{x^4}{12aa} - \frac{x^7}{63a^5} + \frac{x^{10}}{162a^8} \quad \&\text{c} = AFDB.$$

or also

$$r = x + \frac{a^3}{3xx} - \frac{a^6}{9x^5} + \frac{5a^9}{81x^8} \quad \&\text{c}$$

whence

$$z = \frac{xx}{2} - \frac{a^3}{3x} + \frac{a^6}{36x^4} - \frac{5a^9}{567x^7} \quad \&\text{c} = HDBH.$$

[Here Newton gives a second expansion useful when x is large. For example for $\int_0^{1/2} \sqrt[3]{1+x^3}\,dx$ the first four terms of his series give .50509035 while numerical integration on the TI-84 gives .50508999. On the other hand for $\int_{100}^{101} \sqrt[3]{1+x^3}\,dx$ the first 3 terms of his second series give 100.500033, the same approximation as the calculator.]

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